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Proof: $\sum \cup \{\alpha\} \vdash \phi \Rightarrow \sum \vdash (\alpha \to \phi)$ (Deduction Theorem)

The Deduction Theorem is a very useful tool in the work of formal logic. However, the Deduction Theorem is a metatheorem, which is to say it is used to deduce the existence of a proof in a given theory from an already existing proof in the given theory, without belonging to the theory itself. First, a few simple definitions and propositions:

- *Def 1:* $\Rightarrow$ *Implication in metalanguage.*
- *Def 2:* $\to$ *Implication in object language.*
- *Prop 1:* $\beta \in \sum \Rightarrow \sum \vdash \beta$
- *Prop 2:* $\sum \vdash \gamma$ *and* $\gamma \to \alpha \Rightarrow \sum \vdash \alpha$ *(Modus Ponens)*
- *Prop 3:* $\vdash \alpha \to \alpha$
- *Prop 4:* $\vdash \alpha \to \sum \vdash \alpha$*, for any* $\sum$

Since we have $\sum \cup \{\alpha\} \vdash \phi$, we will let $\phi_1, \phi_2, ..., \phi_n$ be a proof of ϕ from $\sum \cup \{\alpha\}$, where $\phi_n = \phi$. We will prove by induction on i that $\sum \vdash (\alpha \to \phi_i)$. First, notice that ϕ_1 must be in 1 of 3 places:

(a)	in $\sum$
(b)	axiom of PC
(c)	α

So, we need to show that for each of these three cases and $i = 1$, $\sum \vdash (\alpha \to \phi_i)$.

$(a1)$	$\phi_1 \to (\alpha \to \phi_1)$	: PC Axiom 1
$(a2)$	$\sum \vdash \phi_1$	: Prop 2
$(a3)$	$\sum \vdash (\alpha \to \phi_1)$ **for case a**	: MP, Prop 1
$(b1)$	$\vdash (\alpha \to \phi_1)$	: MP, PC Axiom
$(b2)$	$\sum \vdash (\alpha \to \phi_1)$ **for case b**	: Prop 4
$(c1)$	$\vdash (\alpha \to \phi_1)$ for case c	: Prop 3
$(c2)$	$\sum \vdash (\alpha \to \phi_1)$ **for case c**	: Prop 4

Thus, for $i = 1$, we have shown that $\sum \vdash (\alpha \rightarrow \phi_i)$. Next comes the induction step. Assume that $\sum \vdash (\alpha \rightarrow \phi_k)$, for all $k < i$. Thus, the next step we haven't shown in our proof, ϕ_i, could be in one of 4 places:

(d)	in $\sum$
(e)	axiom of PC
(f)	α
(g)	follow by MP from some ϕ_j, ϕ_m, where $j < i$, $m < i$, and $\phi_m = \phi_j \rightarrow \phi_i$

Showing that $\sum \vdash (\alpha \rightarrow \phi_i)$ (d), (e), and (f) is done similar to (a), (b), and (c) above. All that is left, is to show $\sum \vdash (\alpha \rightarrow \phi_i)$ for case (g).

$(d1)$	$\sum \vdash (\alpha \rightarrow \phi_i)$ **for cases d, e, f**	: Similar to a, b, c
$(g1)$	$\sum \vdash (\alpha \rightarrow \phi_j)$	: Inductive Hyp.
$(g2)$	$\sum \vdash (\alpha \rightarrow \phi_m)$	: Inductive Hyp.
$(g3)$	$\sum \vdash (\alpha \rightarrow (\phi_j \rightarrow \phi_i))$	: Substitution, g1
$(g4)$	$\sum \vdash ((\alpha \rightarrow (\phi_j \rightarrow \phi_i)) \rightarrow ((\alpha \rightarrow \phi_j) \rightarrow (\alpha \rightarrow \phi_i)))$	: PC Axiom 2
$(g5)$	$\sum \vdash ((\alpha \rightarrow \phi_j \rightarrow (\alpha \rightarrow \phi_i))$	: MP, g3, g4
$(g6)$	$\sum \vdash (\alpha \rightarrow \phi_i)$ **for case g**	: MP, g5, g1

This concludes the inductive step, which shows $\sum \vdash (\alpha \rightarrow \phi_i)$ for all $i > 1$, while the "base" case handles $i = 1$. Letting $i = n$, we get $\sum \vdash (\alpha \rightarrow \phi_n)$, which by substitution results in $\sum \vdash (\alpha \rightarrow \phi)$.

$$\therefore \sum \cup \{\alpha\} \vdash \phi \Rightarrow \sum \vdash (\alpha \rightarrow \phi)$$